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On Convergence of Roe's Scheme for the General Non-linear Scalar Wave Equation

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The convergence of Roe's scheme for the non-linear scalar wave equation to a weak solution of the Cauchy problem is studied and a modification is indicated which makes the scheme entropy satisfying.

1. INTRODUCTION

Recently Le Roux [1] and Sanders [2] have proved the convergence of schemes for the non-linear scalar wave equation

$$u_t + f(u)_x = 0.$$

In this paper we give a similar proof of convergence for the scheme of Roe [3], showing that this scheme converges to a weak solution of the Cauchy problem for general $f(u)$.

Section 2 contains a description of the problem and of the difference scheme. Some preliminary results are stated in Section 3 and the main convergence theorem is proved in Section 4. Entropy violation is discussed in Section 5, and comments are in Section 6.

2. THE PROBLEM AND THE DIFFERENCE SCHEME

(a) *The Problem*

We consider the equation

$$u_t + [f(u)]_x = 0 \tag{2.1}$$

for (x, t) in $\mathbb{R} \times (0, T)$, $T > 0$ and f in $C^1(\mathbb{R})$, with

$$u(x, 0) = u_0(x) \tag{2.2}$$

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for x in $\mathbb{R}$ and u_0 in $L^\infty(\mathbb{R})$, assumed to be of locally bounded variation on $\mathbb{R}$ and therefore satisfying, for all real δ,

$$\forall R \geqslant 0, \qquad \int_{|x|<R} |u_0(x+\delta) - u_0(x)| \, dx \leqslant C(R) \, |\delta|, \tag{2.3}$$

where C is an increasing function on $[0, \infty)$, independent of δ.

The Cauchy problem associated with (2.1) and (2.2) is to find a bounded function u which satisfies (2.1), (2.2). A weak solution to the Cauchy problem is a function u in $L^\infty(\mathbb{R} \times (0, T))$ which satisfies an integral form of (2.1), namely,

$$\iint_{\mathbb{R}\times(0,T)} \left(u \frac{\partial \psi}{\partial t} + f(u) \frac{\partial \psi}{\partial x} \right) dx \, dt + \int_{\mathbb{R}} u_0(x) \, \psi(x, 0) \, dx = 0 \tag{2.4}$$

for all test functions ψ in C^∞ $(\mathbb{R} \times [0, T))$ of compact support in $\mathbb{R} \times [0, T)$.

We consider the approximations generated by the finite difference scheme of Roe [3] and discuss their convergence to a weak solution of the Cauchy problem.

Let Δx be the spatial grid size, with $0 < \Delta x < \Delta x_0$, and Δt be the time grid size, related to Δx by the fixed positive number λ through

$$\lambda = \frac{\Delta t}{\Delta x}. \tag{2.5}$$

In a neighbourhood of the gridpoint $(k \, \Delta x, n \, \Delta t)$ define the rectangle

$$I_k \times J_n = ((k - \tfrac{1}{2}) \, \Delta x, (k + \tfrac{1}{2}) \, \Delta x) \times ((n - \tfrac{1}{2}) \, \lambda \, \Delta x, (n + \tfrac{1}{2}) \, \lambda \, \Delta x) \tag{2.6}$$

for $k \in \mathbb{Z}$, $n \in \mathbb{N}$ and $n \leqslant N = [T/\lambda \, \Delta x] + 1$, where $[y]$ denotes the integer part of y.

We approach a weak solution of (2.1), (2.2) in the sense of (2.4) by a piecewise constant function u_Δ defined on $\mathbb{R} \times (0, T)$ by

$$u_\Delta(x, t) = u_k^n \qquad \text{for} \quad (x, t) \in I_k \times J_n, \tag{2.7}$$

where the initial condition (2.2) is projected onto the space of piecewise constant functions by the restriction

$$u_k^0 = \frac{1}{\Delta x} \int_{I_k} u_0(x) \, dx. \tag{2.8}$$

(b) *The Difference Scheme*

The values u_k^n are calculated as follows (see [3]). (For brevity we write

$$u_k = u_k^n, \qquad u^k = u_k^{n+1} \tag{2.9}$$

as long as there is no danger of confusion.)

Let $\nu_{k-1/2}$ be the approximation

$$\nu_{k-1/2} = \lambda \frac{\delta f_{k-1/2}}{\delta u_{k-1/2}} \tag{2.10}$$

to the CFL number in the cell $I_{k-1/2}$ (see (2.6)) where $f_k = f(u_k)$ and $\delta f_{k-1/2} = f_k - f_{k-1}$. Let also

$$s_{k-1/2} = \operatorname{sgn}(\nu_{k-1/2}) \tag{2.11}$$

be the sign of $\nu_{k-1/2}$, and define

$$\phi_{k-1/2} = -\lambda\, \delta f_{k-1/2} = -\nu_{k-1/2}\, \delta u_{k-1/2} \tag{2.12}$$

to be the flux increment or *fluctuation* in the cell $I_{k-1/2}$.

We obtain a first-order-accurate scheme when the quantity $\phi_{k-1/2}$ is added to the value of u at the downwind end of the cell over the time step Δt. (If $\nu_{k-1/2} = 0$, then $\phi_{k-1/2} = 0$, so no ambiguity arises.) This is the first-order upwinded scheme, which can be represented graphically as shown in Fig. 1.

Now let

$$k' = k - s_{k-1/2}, \qquad \alpha_{k-1/2} = \tfrac{1}{2}(1 - |\nu_{k-1/2}|) \tag{2.13}$$

and define the quantity

$$b_{k-1/2} = \begin{cases} \operatorname{minmod}\{\alpha_{k-1/2}\phi_{k-1/2}, \alpha_{k'-1/2}\phi_{k'-1/2}\}, & \nu_{k-1/2} \cdot \nu_{k'-1/2} \geqslant 0 \quad (2.14a) \\ 0, & \nu_{k-1/2} \cdot \nu_{k'-1/2} < 0, \quad (2.14b) \end{cases}$$

where the operator minmod selects the argument with minimum modulus; (2.14b) corresponds to the positions of shocks or expansions.

If $b_{k-1/2}$ is transferred across the cell *against* the stream direction we generate a scheme which is second-order-accurate at all points except for discontinuities of the solution [9]. The scheme may be identified as either the Lax–Wendroff scheme [4] or the upwind scheme of Warming and Beam [5] depending on the choice in (2.14a), which switches between the two. The transfer of $b_{k-1/2}$ may be regarded as an antidif-

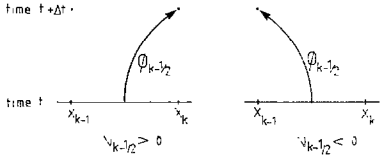

FIG. 1. First-order scheme.

fusion step (see [1]), the complete algorithm (represented graphically in Fig. 2) being written as

$$u^k = u_k + \tfrac{1}{2}(1 - s_{k+1/2})\,\phi_{k+1/2} + \tfrac{1}{2}(1 + s_{k-1/2})\,\phi_{k-1/2} - s_{k-1/2} b_{k-1/2} + s_{k+1/2} b_{k+1/2}, \tag{2.15}$$

where

$$s_{k-1/2} = \operatorname{sgn}(v_{k-1/2}). \tag{2.16}$$

In subsequent work it will be convenient to define two ratios $\beta_{k-1/2}$ and $\beta'_{k-1/2}$, namely,

$$\beta_{k-1/2} = \frac{b_{k-1/2}}{\alpha_{k-1/2}\phi_{k-1/2}}, \qquad \beta'_{k-1/2} = \frac{b_{k-1/2}}{\alpha_{k'-1/2}\phi_{k'-1/2}}, \tag{2.17}$$

which, from the definition (2.14), have the properties

$$|\beta_{k-1/2}| \leqslant 1, \qquad |\beta'_{k-1/2}| \leqslant 1. \tag{2.18}$$

3. Preliminaries

We now prove two lemmas and quote two theorems which will be used in the proof of the main convergence theorem in Section 4. The lemmas are proved for a difference scheme of the general form (3.1) below, which is non-linear through the presence of data-dependent coefficients and includes the scheme in Section 2 as a special case.

Lemma 1. *A difference scheme in the form*

$$u^k = u_k + \xi_{k+1/2}\phi_{k+1/2} + \zeta_{k-1/2}\phi_{k-1/2}, \tag{3.1}$$

where $\phi_{k\pm 1/2}$ is defined by (2.12) *and $\xi_{k+1/2}$, $\zeta_{k-1/2}$ may be data dependent* (*including dependence on the ϕ's*), *satisfies the local bound*

$$\inf\{u_{k-1}, u_k, u_{k+1}\} \leqslant u^k \leqslant \sup\{u_{k-1}, u_k, u_{k+1}\} \tag{3.2}$$

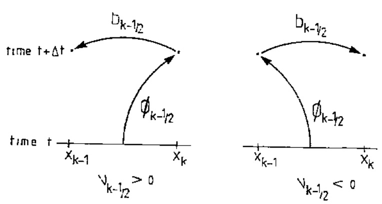

Fig. 2. Second-order scheme.

if the following inequalities are satisfied:

$$\begin{aligned} &0 \leqslant \zeta_{k-1/2} \nu_{k-1/2} \\ &0 \leqslant -\xi_{k+1/2} \nu_{k+1/2} \\ &0 \leqslant -\xi_{k+1/2} \nu_{k+1/2} + \zeta_{k-1/2} \nu_{k-1/2} \leqslant 1. \end{aligned} \tag{3.3}$$

Proof.

$$\begin{aligned} u^k &= u_k + \xi_{k+1/2} \phi_{k+1/2} + \zeta_{k-1/2} \phi_{k-1/2} \\ &= u_{k-1}\{\zeta_{k-1/2} \nu_{k-1/2}\} \\ &\quad + u_k\{1 + \xi_{k+1/2} \nu_{k+1/2} - \zeta_{k-1/2} \nu_{k-1/2}\} + u_{k+1}\{-\xi_{k+1/2} \nu_{k+1/2}\}. \end{aligned} \tag{3.4}$$

If the inequalities (3.3) are satisfied the coefficients of the u's are non-negative and we obtain

$$\begin{aligned} u^k \leqslant &\{\zeta_{k-1/2} \nu_{k-1/2}\} u_{\max} + \{1 + \xi_{k+1/2} \nu_{k+1/2} - \zeta_{k-1/2} \nu_{k-1/2}\} u_{\max} \\ &+ \{-\xi_{k+1/2} \nu_{k+1/2}\} u_{\max}, \end{aligned}$$

from which $u^k \leqslant u_{\max}$, where

$$u_{\max} = \sup\{u_{k-1}, u_k, u_{k+1}\}.$$

Similarly, $u^k \geqslant u_{\min}$, where

$$u_{\min} = \inf\{u_{k-1}, u_k, u_{k+1}\}.$$

This completes the proof.

LEMMA 2. *A difference scheme in the form* (3.1) *conserves local bounded variation in the sense*

$$\sum_{|k| \leqslant K} |u_{k+1}^{n+1} - u_k^{n+1}| \leqslant \sum_{|k| \leqslant K+n} |u_{k+1}^0 - u_k^0| \tag{3.5}$$

for all $K > 0$ *if the following inequalities are satisfied.*

$$\begin{aligned} &0 \leqslant -\xi_{k+1/2} \nu_{k+1/2} \\ &0 \leqslant \zeta_{k+1/2} \nu_{k+1/2} \\ &0 \leqslant (\zeta_{k+1/2} - \xi_{k+1/2}) \nu_{k+1/2} \leqslant 1. \end{aligned} \tag{3.6}$$

Proof.

$$u^{k+1} - u^k = u_{k+1} - u_k + \xi_{k+3/2}\phi_{k+3/2} + (\zeta_{k+1/2} - \xi_{k+1/2})\,\phi_{k+1/2} - \zeta_{k-1/2}\phi_{k-1/2} \tag{3.7}$$

$$= \{-\xi_{k+3/2}\nu_{k+3/2}\}\,\delta u_{k+3/2}$$

$$+ \{1 - (\zeta_{k+1/2} - \xi_{k+1/2})\,\nu_{k+1/2}\}\,\delta u_{k+1/2} + \{\zeta_{k-1/2}\nu_{k-1/2}\}\,\delta u_{k-1/2}. \tag{3.8}$$

Taking absolute values and summing over $|k| \leqslant K$, we obtain

$$\sum_{|k| \leqslant K} |u^{k+1} - u^k| \leqslant \sum_{|k| \leqslant K} \{|-\xi_{k+1/2}\nu_{k+1/2}| + |1 - (\zeta_{k+1/2} - \xi_{k+1/2})\,\nu_{k+1/2}|$$

$$+ |\zeta_{k+1/2}\nu_{k+1/2}|\}\,|u_{k+1} - u_k|$$

$$+ |-\xi_{K+3/2}\nu_{K+3/2}|\,|u_{K+2} - u_{K+1}|$$

$$+ |\zeta_{-K-1/2}\nu_{-K-1/2}|\,|u_{-K} - u_{-K-1}| \tag{3.9}$$

using summation by parts. If the inequalities (3.6) hold we may remove the modulus signs in the coefficients, obtaining

$$\sum_{|k| \leqslant K} |u_{k+1}^{n+1} - u_k^{n+1}| \leqslant \sum_{|k| \leqslant K+1} |u_{k+1}^n - u_k^n|. \tag{3.10}$$

Repeated application gives (3.5) as required.

This completes the proof. We shall show later that the difference scheme of Section 2(b) satisfies the conditions of Lemmas 1 and 2.

We now quote Helly's Theorem (see [6, pp. 29–30]), which will be required in Section 4.

HELLY'S THEOREM. *Let the sequence of functions* $\{g_n(x)\}_0^\infty$ *be of uniformly bounded variation in* $a \leqslant x \leqslant b$ *and such that*

$$|g_n(a)| < A \qquad (n = 0, 1, 2, \ldots)$$

for some constant A. *There then exists a set of integers*

$$n_0 < n_1 < n_2 < \cdots$$

and a function $g(x)$ *of bounded variation in* $a \leqslant x \leqslant b$ *such that*

$$\lim_{i \to \infty} g_{n_i}(x) = g(x) \qquad (a \leqslant x \leqslant b).$$

That is, given a sequence of functions which are uniformly bounded and of uniformly bounded variation on an interval, it is possible to extract a subsequence which converges to a function of bounded variation in L^1.

Finally, we quote the Lax–Wendroff Theorem for difference schemes written in conservation form [4], which will also be used in Section 4.

THE LAX–WENDROFF THEOREM. *Consider a difference scheme of the form*

$$u^k = u_k - \lambda(h_{k+1/2} - h_{k-1/2}),$$

where

$$h_{k+1/2} = h(u_{-k+1},\dots, u_k)$$

and, for consistency with (2.1),

$$h(u,\dots, u) = f(u).$$

Suppose that, as Δx, Δt tend to zero, the solution $v(x, t)$ produced by the scheme, if applied at every x, converges boundedly almost everywhere to some function $u(x, t)$. Then $u(x, t)$ is a weak solution of (2.1) *with initial data* (2.2).

The convergence theorem follows.

4. CONVERGENCE OF THE DIFFERENCE SCHEME

We now state and prove our main theorem.

THEOREM. *Suppose that u_0 lies in $L^\infty(\mathbb{R}) \cap BV_{\mathrm{loc}}(\mathbb{R})$ and that the condition*

$$\sup_k |v_k| \leqslant 1 \tag{4.1}$$

is satisfied. Then the family of approximations $\{u_\Delta\}$ generated by the difference scheme (2.15) *from initial data* (2.8) *contains a subsequence $\{u_{\Delta_m}\}$ which converges in $L^1_{\mathrm{loc}}(\mathbb{R} \times (0, T))$ towards a weak solution of* (2.1), (2.2) *as $\Delta x_m \to 0$.*

Proof. The proof is in three main parts. First we show that the piecewise constant function (2.7) generated by the scheme of Section 2(b) is uniformly bounded and of uniformly bounded variation in space and time. Then we demonstrate that from the family of such functions we can extract a sequence convergent in $L^1_{\mathrm{loc}}(\mathbb{R} \times (0, T))$. Finally, we prove that the limit function is in fact a weak solution of the problem.

The scheme may be written in the form

$$u^k = u_k + \xi_{k+1/2}\phi_{k+1/2} + \zeta_{k-1/2}\phi_{k-1/2}, \tag{4.2}$$

where $\xi_{k+1/2}, \zeta_{k-1/2}$ are given by

$$\zeta_{k-1/2} = \begin{cases} 1 + (b_{k+1/2} - b_{k-1/2})/\phi_{k-1/2}, & v_{k\pm 1/2} > 0 \\ 0, & v_{k-1/2} < 0 \\ 1, & v_{k+1/2} < 0,\ v_{k-1/2} > 0 \end{cases} \tag{4.3}$$

$$\xi_{k+1/2} = \begin{cases} 0, & v_{k+1/2} > 0 \\ 1 - (b_{k+1/2} - b_{k-1/2})/\phi_{k+1/2}, & v_{k\pm 1/2} < 0 \\ 1, & v_{k+1/2} < 0,\ v_{k-1/2} > 0. \end{cases} \tag{4.4}$$

(Note that each term $\xi_{k+1/2}\phi_{k+1/2}$ and $\zeta_{k-1/2}\phi_{k-1/2}$ in (4.2) may generate a scheme with three-point support. Generally one term or the other will be zero: only at a shock will both be non-zero.)

Using the definitions of α, β, β' in (2.13) and (2.17), these become

$$\zeta_{k-1/2} = \begin{cases} 1 + (\beta'_{k+1/2} - \beta_{k-1/2})\,\alpha_{k-1/2}, & \nu_{k\pm 1/2} > 0 \\ 0, & \nu_{k-1/2} < 0 \\ 1, & \nu_{k+1/2} < 0,\ \nu_{k-1/2} > 0 \end{cases} \tag{4.5}$$

$$\xi_{k+1/2} = \begin{cases} 0, & \nu_{k+1/2} > 0 \\ 1 + (\beta'_{k-1/2} - \beta_{k+1/2})\,\alpha_{k+1/2}, & \nu_{k+1/2} < 0 \\ 1, & \nu_{k+1/2} < 0,\ \nu_{k-1/2} > 0. \end{cases} \tag{4.6}$$

We have from (2.18) and (2.13) the bounds

$$-2 \leqslant (\beta'_{k\pm 1/2} - \beta_{\mp 1/2}) \leqslant 2 \tag{4.7}$$

and

$$0 \leqslant \alpha_{k\pm 1/2} \leqslant \tfrac{1}{2} \tag{4.8}$$

from which we deduce, using (4.5) and (4.6), that

$$\begin{aligned} 0 &\leqslant -\xi_{k+1/2}\nu_{k+1/2} \\ 0 &\leqslant \zeta_{k-1/2}\nu_{k-1/2}. \end{aligned} \tag{4.9}$$

Consider now the expression

$$-\xi_{k+1/2}\nu_{k+1/2} + \zeta_{k-1/2}\nu_{k-1/2} \tag{4.10}$$

which occurs in the inequalities (3.3) of Lemma 2. If $\nu_{k\pm 1/2}$ are of the same sign we have (taking the positive sign as an example)

$$\begin{aligned} -\xi_{k+1/2}\nu_{k+1/2} + \zeta_{k-1/2}\nu_{k-1/2} &= (1 + (\beta'_{k+1/2} - \beta_{k-1/2})\,\alpha_{k-1/2})\,\nu_{k-1/2} \\ &\leqslant (1 + 2\alpha_{k-1/2})\,\nu_{k-1/2} \qquad \text{(from (4.7))} \\ &= (2 - |\nu_{k-1/2}|)\,\nu_{k-1/2} \\ &\leqslant 1 \end{aligned} \tag{4.11}$$

by condition (4.1). If $\nu_{k+1/2}, \nu_{k-1/2}$ are of opposite sign then there are two cases to consider. For an expansion wave, i.e., $\nu_{k+1/2} > 0$, $\nu_{k-1/2} < 0$, then $\xi_{k+1/2} = \zeta_{k-1/2} = 0$ so that trivially

$$-\xi_{k+1/2}\nu_{k+1/2} + \zeta_{k-1/2}\nu_{k-1/2} \leqslant 1, \tag{4.12}$$

whilst for a compression wave (shock), i.e., $\nu_{k+1/2} < 0$, $\nu_{k-1/2} > 0$, (4.10) becomes

$$\begin{aligned} -\xi_{k+1/2}\nu_{k+1/2} + \zeta_{k-1/2}\nu_{k-1/2} &= \nu_{k-1/2} - \nu_{k+1/2} \\ &\leqslant 2 \end{aligned} \tag{4.13}$$

by condition (4.1).

Consider next the expression

$$(\zeta_{k+1/2} - \xi_{k+1/2})\,\nu_{k+1/2} \tag{4.14}$$

which occurs in the inequalities (3.6) of Lemma 2. From (4.5), (4.6) we have

$$\begin{aligned} \zeta_{k+1/2} &= 0 \qquad \text{if} \quad \nu_{k+1/2} < 0 \\ \xi_{k+1/2} &= 0 \qquad \text{if} \quad \nu_{k+1/2} > 0 \end{aligned} \tag{4.15}$$

so that using (4.11), we obtain

$$(\zeta_{k+1/2} - \xi_{k+1/2})\,\nu_{k+1/2} \leqslant 1. \tag{4.16}$$

Now for the cases covered by (4.9), (4.11) and (4.12) we see that the conditions of Lemma 1 are satisfied, and hence that

$$\inf\{u_{k-1}, u_k, u_{k+1}\} \leqslant u^k \leqslant \sup\{u_{k-1}, u_k, u_{k+1}\}. \tag{4.17}$$

For the case of the compression, represented by Eq. (4.13), the conditions of Lemma 1 are not satisfied: however, we are indebted to A. Y. LeRoux for bringing to our attention a direct proof of (4.17) in this instance, which may be found in [14]. Finally, by induction, we may readily deduce that

$$\|u_k\|_{L^\infty(\mathbb{R}\times[0,T))} \leqslant \|u_0\|_{L^\infty(\mathbb{R})}. \tag{4.18}$$

Also from (4.9) and (4.16) the conditions of Lemma 2 are met and hence

$$\sum_{|k|\leqslant K} |u_{k+1}^{n+1} - u_k^{n+1}| \leqslant \sum_{|k|\leqslant K+n} |u_{k+1}^0 - u_k^0| \tag{4.19}$$

for all $K > 0$.

Choose $R > 0$ and set $K = [R/\Delta x]$: then, using (2.3), (4.19) becomes

$$\begin{aligned} \sum_{|k|\leqslant K} |u_{k+1}^{n+1} - u_k^{n+1}| &\leqslant \frac{1}{\Delta x}\int_{|x|\leqslant R+T/\lambda} |u_0(x+h) - u_0(x)|\,dx \\ &\leqslant C(R + T/\lambda), \end{aligned} \tag{4.20}$$

where $C(R + T/\lambda)$ is a constant depending only on the region Ω_R defined by

$$\Omega_R = (-R, R) \times (0, T). \tag{4.21}$$

Summarizing, we have shown that Roe's scheme generates the family of functions $\{u_\Delta(x, t)\}$ (see (2.7)) with the following properties:

(a) $u_\Delta(x, t) = u_k^n$ in the rectangle $I_k \times J_n$ (see (2.6)).

(b) $u_\Delta(x, t)$ is uniformly bounded by $\| u_0 \|_{L^\infty(\mathbb{R})}$, from (4.18).

(c) $u_\Delta(x, t)$ is of uniformly bounded variation in the x coordinate, from (4.20).

(d) $u_\Delta(x, t)$ is of uniformly bounded variation in the time coordinate, since, from (2.14), (2.15) and (2.16),

$$|u_k^{n+1} - u_k^n| \leqslant \max\{|u_{k+1}^n - u_k^n|, |u_k^n - u_{k-1}^n|\}, \tag{4.22}$$

so that, from (b) above there is also a bound on the time variation of $u_\Delta(x, t)$. This completes the first part of the proof.

Now, following Oleinik [7], let $t = t_m$ $(m = 1, 2,...)$ be a countable everywhere dense set on the segment $[0, T]$ in Ω_R. By Helly's Theorem, (see Section 3), on any straight line $t = \text{constant} > 0$ we can extract from $\{u_\Delta\}$ a subsequence, converging at every point of this straight line as $\Delta x \to 0$.

Hence on the line $t = t_1$ we extract a sequence $\{u_{\Delta_1}\}$ from $\{u_\Delta\}$, then on the line $t = t_2$ we extract from $\{u_{\Delta_1}\}$ a subsequence $\{u_{\Delta_2}\}$ and so on. Then, by means of a diagonal process (see [8, pp. 301]), which consists of taking the ith element of the ith sequence, we can extract a sequence $\{u_\Delta^i\} = \{u_{\Delta_i}^i\}$ $(i \to \infty, \Delta_x \to 0)$ which converges at every point of the family of straight lines $t = t_m$ $(m = 1, 2,...)$ for $i \to \infty$.

We now show that $\{u_\Delta^i\}$ is Cauchy in $L^1(\Omega_R)$, i.e.,

$$\int_{|x|<R} |u_\Delta^i(x, t) - u_\Delta^j(x, t)|\, dx \to 0 \qquad \text{as} \quad i, j \to \infty, \forall\, t. \tag{4.23}$$

Since u_Δ is constant on $((k - \frac{1}{2})\Delta x, (k + \frac{1}{2})\Delta) \times ((n - \frac{1}{2})\, \lambda\, \Delta x, (n + \frac{1}{2})\, \lambda\, \Delta x)$ we have $u_\Delta(x, t) = u_\Delta(x, n\lambda\, \Delta x)$, where $n = [t/\lambda\, \Delta x + \frac{1}{2}]$ and $[y]$ again denotes the integer part of y.

Since the set $t = t_m$ $(m = 1, 2,...)$ is everywhere dense we can choose from it a sequence $\{t_{m_s}\}$ converging to t as $m_s \to \infty$. Setting $n_s = [t_{m_s}/\lambda\, \Delta x + \frac{1}{2}]$, we have

$$\begin{aligned} &\int_{|x|<R} |u_\Delta^i(x, t) - u_\Delta^j(x, t)|\, dx \\ &\quad \leqslant \int_{|x|<R} |u_\Delta^i(x, n\lambda\, \Delta x) - u_\Delta^i(x, n_s\lambda\, \Delta x)|\, dx \\ &\quad\quad + \int_{|x|<R} |u_\Delta^j(x, n\lambda\, \Delta x) - u_\Delta^j(x, n_s\lambda\, \Delta x)|\, dx \\ &\quad\quad + \int_{|x|<R} |u_\Delta^i(x, n_s\lambda\, \Delta x) - u_\Delta^j(x, n_s\lambda\, \Delta x)|\, dx. \end{aligned} \tag{4.24}$$

The first term on the right-hand side of (4.24) is bounded by

$$\sum_{|k| \leqslant K} |u_k^n - u_k^{n_s}| \, \Delta x$$

since $K = [R/\Delta x]$, which in turn is bounded by

$$\sum_{|k| \leqslant K} \sum_{n=n_1}^{n_2-1} |u_k^{n+1} - u_k^n| \, \Delta x, \tag{4.25}$$

writing n_1, n_2 for the minimum and maximum of n, n_s, respectively. Now from (4.22)

$$\begin{aligned}
\sum_{|k| \leqslant K} \sum_{n_1}^{n_2-1} |u_k^{n+1} - u_k^n| \, \Delta x &\leqslant 2 \sum_{n_1}^{n_2-1} \sum_{|k| \leqslant K} |u_{k+1}^n - u_k^n| \, \Delta x \\
&\leqslant 2 \sum_{n_1}^{n_2-1} C(R + T/\lambda) \, \Delta x \qquad \text{(from (4.20))} \\
&\leqslant 2(n_2 - n_1) \, C(R + T/\lambda) \, \Delta x \\
&= (2/\lambda) \, |t - t_{m_s}| \, C(R + T/\lambda) \\
&\to 0 \qquad \text{as} \quad t_{m_s} \to t.
\end{aligned} \tag{4.26}$$

Thus the first term on the right-hand side of (4.24) $\to 0$ as $t_{m_s} \to t$ and the same is true for the second term.

Since the sequence t_{m_s} has been chosen from the set $t = t_m$ $(m = 1, 2, \ldots)$ and since the sequence $\{u_\Delta^i\}$ is convergent on each line $t = t_m$ it is also convergent on $t = t_{m_s}$ and thus is Cauchy on $t = t_{m_s}$. Hence the last term in (4.24) $\to 0$ as $i \to \infty$, $j \to \infty$. Thus we have proved (4.23) and shown that the sequence $\{u_\Delta^i\}$ converges to a function $u(x, t)$ in $L^1(\Omega_R)$.

We have therefore obtained a sequence $\{u_{\Delta_{R_i}}^i\}$ from $\{u_\Delta\}$ converging in $L^1(\Omega_R)$, and similarly we may obtain, from $\{u_{\Delta_R}\}$, a sequence $\{u_{\Delta_{R+1}}^i\}$ converging in $L^1(\Omega_{R+1})$ and so on. Then by a diagonal process (see above) we may obtain a sequence $\{u_{\Delta_{R+m}}^m\}$ extracted from $\{u_\Delta\}$ which converges in $L_{\text{loc}}^1(\mathbb{R} \times (0, T))$ to $u(x, t)$. It is evident that $u(x, t) \in L^\infty(\mathbb{R} \times (0, T))$.

This completes the second part of the proof.

It remains to show that u is a weak solution of (2.1), (2.2). To do this we rewrite (2.15) in conservation form,

$$\begin{aligned}
u^k &= u_k + \tfrac{1}{2}(1 - s_{k+1/2}) \, \phi_{k+1/2} + \tfrac{1}{2}(1 + s_{k-1/2}) \, \phi_{k-1/2} + s_{k+1/2} b_{k+1/2} - s_{k-1/2} b_{k-1/2} \\
&= u_k + \tfrac{1}{2}(\phi_{k+1/2} + \phi_{k-1/2}) + s_{k+1/2}(b_{k+1/2} - \tfrac{1}{2}\phi_{k+1/2}) - s_{k-1/2}(b_{k-1/2} - \tfrac{1}{2}\phi_{k-1/2}) \\
&= u_k - \tfrac{1}{2}\lambda(f_{k+1} - f_{k-1}) + s_{k+1/2}(b_{k+1/2} - \tfrac{1}{2}\phi_{k+1/2}) - s_{k-1/2}(b_{k-1/2} - \tfrac{1}{2}b_{k-1/2}) \\
&= u_k - \lambda\{[\tfrac{1}{2}(f_{k+1} + f_k) - (1/\lambda) \, s_{k+1/2}(b_{k+1/2} - \tfrac{1}{2}\phi_{k+1/2})] \\
&\quad - [\tfrac{1}{2}(f_k - f_{k-1}) - (1/\lambda) \, s_{k-1/2}(b_{k-1/2} - \tfrac{1}{2}\phi_{k-1/2})]\},
\end{aligned} \tag{4.27}$$

i.e.,

$$u^k = u_k - \lambda(h_{k+1/2} - h_{k-1/2}), \tag{4.28}$$

where

$$\begin{aligned} h_{k+1/2} &= h(u_{k+2},\ldots, u_{k-1}) \\ &= \tfrac{1}{2}(f_{k+1} + f_k) - (1/\lambda)\, s_{k+1/2}(b_{k+1/2} - \tfrac{1}{2}\phi_{k+1/2}). \end{aligned} \tag{4.29}$$

Note that

$$\begin{aligned} h(u,\ldots, u) &= \tfrac{1}{2}(f(u) + f(u)) - 0 \\ &= f(u). \end{aligned}$$

Thus the scheme satisfies the conditions of the Lax–Wendroff Theorem (see Section 3) and, since $u_\Delta(x, t)$ converges boundedly to $u(x, t)$, this theorem shows that $u(x, t)$ is a weak solution of the problem (2.1), (2.2).

This completes the proof of the convergence theorem.

5. Entropy Considerations

Although we have shown convergence of Roe's scheme (2.15) to a weak solution of the problem (2.1), (2.2) such a solution is not unique [12, 7, 9], and the scheme may in rare situations generate non-physical or entropy-violating solutions. One example is when the initial data are such that

$$u_k^0 = \begin{cases} -1, & k \leqslant j \\ 1, & k > j \end{cases} \tag{5.1}$$

for some j and $f(u) = \frac{1}{2}u^2$ in (2.1) (the inviscid Burger's equation). Then, since $f_k = \frac{1}{2}\,\forall k$ Roe's scheme leaves the initial data unchanged because it relies on differences of f_k's to compute the increments, i.e.,

$$\phi_{k-1/2} = -\lambda(f_k - f_{k-1}) = -\lambda(\tfrac{1}{2} - \tfrac{1}{2}) = 0,$$

even at the discontinuity. (It should be noted that it is also this same property of the scheme that gives sharp steady shocks.)

It is apparent, therefore, that any modification to the scheme that will enable it to disperse such entropy-violating solutions as (5.1) will need more information than just the nodal flux differences. One approach [9] is to regard the zero increment $\phi_{j+1/2}$ in the cell $I_{j+1/2}$ as being due to the cancellation of left-moving and right-moving increments within that cell.

By choosing a suitable value [13, 9], intermediate to u_k and u_{k-1}, it has been found possible to define such left-moving and right-moving increments [9].

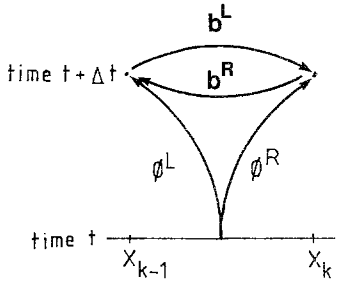

FIG. 3. Modified second-order scheme.

Corresponding left-moving and right-moving transfers can then be constructed, leading to a monotonicity-preserving second-order scheme which disperses entropy-violating solutions. Except in entropy-violating situations, one or other of the left- and right-moving increments is zero and the scheme reduces to Roe's scheme (2.15). A diagram of the scheme is given in Fig. 3.

This modified scheme has been shown [9] to converge to a weak solution.

6. CONCLUSIONS

We have proved that the approximations generated by Roe's scheme (2.15) converge to a weak solution of the problem in Section 2 and have indicated a way in which the scheme may be modified to disperse entropy-violating shocks.

Note that (4.17) demonstrates the important property of monotonicity preservation, that is, monotone data remains monotone after a time step. It is this property of Roe's scheme [3] which has been found particularly valuable in eliminating unwanted oscillations in shock problems; it is here proved for the first time for general $f(u)$.

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